

Constructing Nongeometric Vacua In String Theory

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In this paper we investigate compactifications of the type II and heterotic string on four-dimensional spaces with nongeometric monodromies. We explicitly construct backgrounds which contain the “Duality Twists” discussed by Dabholkar and Hull [1]. Similar constructions of nongeometric backgrounds have been discussed for type II strings by Hellerman, McGreevy, and Williams [2]. We find that imposing such monodromies projects out many moduli from the resulting vacua and argue that these backgrounds are the spacetime realizations of interpolating asymmetric orbifolds.

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1. String theory and Geometry

String theory, in spite of its aspirations as a fundamental theory of quantum gravity, is for the most part heavily reliant on classical notions of geometry. This being said, it is certainly true that strings and point particles probe classical geometries in dramatically different ways. It is well-known that strings can resolve many of the singularities that plague classical and quantum gravity. T-duality establishes a remarkable equivalence between strings compactified on large tori with those compactified on small tori. In more general compactifications, one may take this a step further and relate geometries of different topologies á la mirror symmetry. Each of these ideas, however, still has as its foundation classical geometry. T-duality and mirror symmetry, though they relate very different backgrounds, still serve as relations between two classical geometries, and the stringy resolutions of singularities are understood in the context of strings propagating in classical backgrounds.

There is at least one well-established background which is intrinsically different from ordinary geometry: One may consider “asymmetric orbifolds,” in which the left-moving modes of the string see a different geometry than do the right-moving modes. In such scenarios, one is no longer able to speak of geometry in a meaningful way; this is one example of a nongeometric compactification.

One would like to develop a more general framework in which to discuss string theory backgrounds. This framework should contain both geometric and nongeometric compactifications, and be intimately related to how strings (rather than point particles) probe their background. Such a construction may in fact be a necessary step towards a quantum theory of gravity, since one will no doubt have to replace classical notions of geometry with a quantum alternative. As is often the case in physics, symmetries provide us with an important clue of how to proceed: To construct such nongeometric theories, one may take a hint from stringy symmetries, such as T-duality, which do not exist in ordinary quantum field theory.

T-duality is particularly interesting since, for generic backgrounds, it mixes the metric and B-field [3]. This mixing is an indication that from a string’s perspective the metric and B-field should not be treated as distinct objects but rather as single field. Mixing between the metric and B-field is important for the type of nongeometric compactifications we will investigate in this work; indeed, this is one sense in which a compactification may be regarded as intrinsically nongeometric. Combining the metric and B-field into a single field

is certainly not a new idea; for example, the complexification of the Kähler form arises naturally in string theory. This complexification allows for mirror symmetry and leads to interesting physics such as the flop transition [4]. It is often the case, however, that the metric and B-field are still treated as distinct objects which may be combined into a single field. As will be emphasized throughout this paper, though this field may always be decomposed into symmetric and antisymmetric parts, as one moves around nontrivial paths the components of these parts can become intertwined. By only considering backgrounds where the metric and B-field are distinct objects, one misses a very large class of compactifications.

In the context of the type II string, the work [2] of Hellerman, McGreevy and Williams (HMW) exploits stringy symmetries in order to construct a new class of nongeometric backgrounds. Related ideas, which geometrize the U-duality group, have also been discussed in the context of U-manifolds [5]. The main focus of HMW is T^2 fibrations over an S^2 where the moduli of the fiber undergo nontrivial monodromies. The construction of these nongeometric spaces exactly parallels constructions of geometric spaces which exploit the use of geometric symmetry groups; a prominent example is the construction of K3 via a torus fibration where the fiber has nontrivial twists by the (geometric) modular group of the torus [6]. This construction of K3 has played a central role in F-theory descriptions of type IIB string theory [7]. The key difference between the construction of nongeometric backgrounds in [2] and the analogous construction of K3 is that HMW require that the fiber moduli undergo monodromies in both the geometric and nongeometric subgroups of the full perturbative duality group. The boundary conditions imposed by HMW force the moduli of the theory to vary over spaces where the B-field and metric are treated on equal footing and cannot be disentangled. Asymmetric orbifolds are found to be particular limits of these more general nongeometric spaces.

Because the base space in [2] is an S^2 , it does not contain any nontrivial 1-cycles. In turn, the nontrivial monodromies require the existence of singularities in the base. If the base were nonsingular, any closed loop could be shrunk to zero size and the monodromies would be forced to be trivial. Above each of these singularities is a degenerate fiber. It should be noted that although the fibration picture contains singularities, the total space is smooth. Had one started with a base that contained nontrivial 1-cycles, it would no longer be necessary to have a fibration in which the fibers degenerate. Indeed, since there would now be non-contractible loops, it would be possible to have nontrivial monodromies without inducing singularities on the base; such compactifications are the primary focus

of this paper. In particular, we concentrate on T^2 fibrations over a T^2 base. As will be discussed, the requirement that the base be a T^2 forces the moduli of the fiber to lie at fixed points of the imposed monodromy. It follows that many of the moduli in these compactifications are fixed; in particular, it is possible to fix both the complex structure modulus τ and Kähler modulus ρ of the fiber torus.

HMW focused exclusively on the type II case, and in this work we extend the construction of nongeometric theories to the heterotic string. In this case, the modular group is significantly more complicated than that of the type II theories, due to the presence of Wilson lines. This greatly enlarges the number of possible monodromies, making a general analysis quite difficult. As a prototype example, we choose to impose the monodromy equivalent to T-duality along the fiber torus.

A natural question to now ask is, having fixed the geometric moduli of the theory, in what sense is the resulting theory nongeometric? Although the answer may at first sound like a matter of semantics, we believe that this question hits the very heart of what it means to be in a nongeometric background. By imposing boundary conditions (i.e. monodromies) which are intrinsically nongeometric (that is, they mix the metric and the B-field), we arrive at a six-dimensional theory with fewer moduli than the corresponding geometric theory. In the corresponding geometric theory, there exist massless fields corresponding to fluctuations of the (say) Kähler modulus of the torus fiber. In a nongeometric theory, this massless field is removed from the spectrum, and it is in this sense that the resulting theory is nongeometric. In other words, to ask whether or not the ten dimensional theory is geometric or nongeometric is not the appropriate question. Rather, one should ask whether or not the effective six dimensional theory could be obtained from a geometric theory with geometric boundary conditions.

We re-emphasize this point by stressing that a compactification of string theory should in fact specify two different things: The compact manifold **and** the boundary conditions around nontrivial cycles. Such boundary conditions (on the bosonic fields) are usually taken to be periodic, but this is not required. A nongeometric compactification is then a compactification with nongeometric boundary conditions.

It is worth noting that similar compactifications were discussed from a more abstract perspective in [1]. There, the authors use the Scherk-Schwarz method to prove that compactifications which include twists of the U-duality group must lie at fixed points; they also discuss how such theories should have asymmetric orbifold worldsheet descriptions. By explicitly constructing such a theory (from the spacetime perspective), our results shed

light on this result. In particular, we find that the supergravity equations of motion require that the moduli lie at fixed points; this condition is not *a priori* obvious from the supergravity perspective. In addition, we discuss how finding an asymmetric orbifold description of such a theory is a nontrivial procedure, and how tightly modular invariance and level matching constrain the possible descriptions.

The outline of this paper is as follows: In Section 2, we review the work of HMW. We focus specifically on their six dimensional theories coming from nongeometric compactifications, paying special attention to the asymmetric orbifold descriptions of these theories. In Section 3, we consider type II string theory on nongeometric spaces which are T^2 fibrations over a T^2 base. As discussed above, forcing the base to be a T^2 ends up requiring that certain moduli be fixed. In Section 4, we consider a similar background for the heterotic string. The situation is more complicated than the analogous type II theory due to the presence of Wilson lines. However, as in the type II case, one finds that nontrivial monodromies must have fixed points. We pick several examples of such monodromies, and solve for the allowed backgrounds. In Section 5, we discuss interpolating asymmetric orbifolds, and conjecture that these may correspond to the theories found in Sections 3 and 4. Finally, in Section 6, we make some concluding remarks and suggest possibilities for future work.

2. HMW Review: Geometric Constructions of Nongeometric String Theories

2.1. T^2 fibers over an S^2 base

The authors of [2] construct backgrounds by patching together spaces which are locally fibrations of T^2 over \mathbf{R}^2 , such that the total space is a nongeometric T^2 fibration over a spherical base. We now review this construction in detail.

Consider the dimensional reduction of type II string theory on a T^2 . The perturbative duality group of this theory is

$$\mathcal{G}_{II} \equiv O(2, 2; \mathbb{Z}) \sim SL(2, \mathbb{Z})_\tau \times SL(2, \mathbb{Z})_\rho . \quad (2.1)$$

Here τ refers to the complex structure of the torus,

$$ds^2 = \frac{V}{\tau_2} |\tau d\theta_1 + d\theta_2|^2 ; \quad \tau = \tau_1 + i\tau_2 , \quad (2.2)$$

and ρ is the complexified Kähler modulus which combines the B-field with the volume modulus of the torus,

$$\rho = B_{12} + iV. \quad (2.3)$$

$SL(2, \mathbb{Z})_\tau$ is the geometric modular group which identifies the modular parameters defining equivalent tori, while $SL(2, \mathbb{Z})_\rho$ is generated by T-dualities (and a rotation) and shifts in the B-field. As is well-known, one may change the modulus τ by any $SL(2, \mathbb{Z})$ transformation and get a torus physically equivalent to the original. Thus, when building compact spaces which are fibrations of a T^2 , one may construct closed loops in the base such that upon traversing the loop, τ undergoes an $SL(2, \mathbb{Z})$ transformation. In this case, one says that τ has a nontrivial monodromy along the closed loop. A similar story is true for ρ ; that is, the complexified Kähler modulus may have nontrivial monodromy as well. However, in this case, the $SL(2, \mathbb{Z})$ group comes from the stringy symmetry operations of T-duality and shifts in the B-field.

Generically, one expects a T^2 fibration to contain degenerate fibers [8]. It is well-known that there are only a finite number of different possible degenerations, and each of these induces a different monodromy on the moduli. These were classified by Kodaira in [9], but we do not review the classification here; we will instead quote the relevant results when needed.

The construction in [2], which is closely related to [6,7], takes a more active perspective. Consider a T^2 fibered over \mathbf{R}^2 . By imposing a particular monodromy around some closed loop, a singularity is then induced on the base below a degenerate fiber. Conceptually, it is clear that such a singularity exists: When considering monodromies around trivial loops one may always shrink the loop to zero size, which implies that the monodromy is also trivial. Therefore, since \mathbf{R}^2 has no non-trivial cycles, requiring solutions to have non-trivial monodromy forces the existence of singularities. The supergravity description breaks down near these singularities. However, thanks to Kodaira's classification, one is able to describe the physics near the degenerate fibers in terms of strings on A-D-E orbifolds. To construct their backgrounds, HMW assume that far away from the singularities the space is fiberwise a solution to the supergravity equations of motion, and that these regions (where supergravity is valid) may be patched together smoothly with the appropriate A-D-E orbifolds located at degenerate fibers. Though reasonable, one may question the consistency of such a construction. There is a nontrivial check which one may perform by counting the number of hyper, vector, and tensor multiplets and verifying that they obey

$n_H - n_V + 29n_T = 273$, which is necessary for the theory to be anomaly free. This is indeed satisfied by the nongeometric construction of HMW.

The appendices of [2] discuss the constraints coming from Type II supergravity in great detail. In appendix B we rederive (for a slightly different setup) many of these results. For now, we will simply state some of the results which come from requiring a supersymmetric vacuum, i.e. from the condition that all SUSY variations of fermions vanish. The constraint arising from the variation of the dilatino reduces to

$$\frac{\bar{\partial}B_{12}}{V} = i\chi_6\bar{\partial}\Phi \quad , \quad (2.4)$$

where Φ is the dilaton and χ_6 is the six-dimensional chirality of the conserved spinor. From the variation of the gravitino along the fiber one finds that ρ and τ must be holomorphic,

$$\bar{\partial}\tau = 0 \quad \text{and} \quad \bar{\partial}\rho = 0.$$

Generically such solutions preserve (1,0) supersymmetry in six dimensions. For the type IIA theory, solutions with constant ρ preserve (1,1) supersymmetry and solutions with constant τ preserve (2,0) supersymmetry. This is easy to motivate: In general, (2.4) fixes the six dimensional chirality, but for constant ρ the six dimensional chirality is left unfixed. We recall here that a well-known example of such a background arises from compactifying type IIA on a K3 surface, which does indeed preserve (1,1) SUSY. We can now argue that compactifications with constant τ preserve (2,0) SUSY by starting with constant ρ and T-dualizing.

After going to conformal gauge on the base, $ds_{base}^2 = e^{2\phi}|dz|^2$, the variation of the gravitino along the fiber reduces to

$$\partial\bar{\partial}(\phi - \frac{1}{2}\ln\tau_2 - \frac{1}{2}\ln\rho_2) = 0 \quad . \quad (2.5)$$

It is easy to show that the modular invariant solution to (2.5) is

$$e^{2\phi} = \tau_2 \left| \frac{\eta(\tau)^2}{\Pi(z - z_i)^{1/12}} \right|^2 \rho_2 \left| \frac{\eta(\rho)^2}{\Pi(z - \tilde{z}_i)^{1/12}} \right|^2 \quad , \quad (2.6)$$

where η is the Dedekind eta function. The factors of $(z - z_i)$ and $(z - \tilde{z}_i)$ are included so the conformal factor does not vanish at the points z_i (\tilde{z}_i) around which there are τ (ρ) monodromies.

Far away from any of the degenerations, $ds_{base}^2 \sim |z^{-N/12} dz|^2$, where N is the total number of degenerate fibers. For $N = 24$ the metric is $ds_{base}^2 \sim |dz/z^2|^2 = |dz'|^2$; $z' \equiv 1/z$. It is well known that a two dimensional space which has two coordinate patches and a transition function $z' = 1/z$ is a sphere. In other words, the degenerate fibers backreact on the base and cause it to bend, and by including exactly the right number of singularities, the base curls into a sphere. One can show that although the space is compact for $N > 12$, requiring that (2.4) be satisfied at infinity requires that $N = 24$.

It should be noted that the four-dimensional spaces constructed in this manner are non-singular. The “singularities” on the base are not singularities in the whole space, but rather an artifact of describing the space as a T^2 fibration.

Constructing compact spaces in this manner is not a new idea. For example, one may describe K3 as a T^2 fibration by allowing for degenerations which cause τ to undergo monodromies. This description of K3 has played a central role in understanding compactifications of F-theory (see e.g. [7] for discussion). Since τ degenerations come from the geometric symmetries of the torus, the resulting compactification must be purely geometric. This is certainly the case for K3. Similarly, one may construct compact spaces by considering only ρ degenerations. Since T-duality exchanges ρ and τ , these compactifications may also be given a geometric description. HMW concentrate on a class of models in which half of the degenerations induce τ monodromies and half of the degenerations induce ρ monodromies. Since the total order of degenerations must be 24, the authors of [2] refer these backgrounds as $12 + 12'$ models. Because the $12 + 12'$ models have both τ and ρ degenerations within the same compact space they do not admit a geometric description.

Let us now try to gain some intuition about ρ degenerations. Note that under the monodromy $\rho \rightarrow \rho + 1$ there is not any mixing of the metric and the B-field. In turn, these degenerations do not require the existence of a new nongeometric background. Rather, they can be described by a geometric background in the presence of stringy objects. As discussed in [2], these degenerations correspond to NS5-branes. This is consistent with the fact that constant τ solutions in IIA generically preserve (2,0) supersymmetry. Under T-duality, NS5-branes are exchanged with Kaluza-Klein monopoles; the monodromy around a KK monopole is $\tau \rightarrow \tau + 1$, as expected. This offers an interesting geometrical picture of the degenerate fiber: It is well known that the KK monopole arises from the dimensional reduction of Taub-NUT space. Taub-NUT space is often described as the surface of a “half cigar.” The degenerate fiber around which $\tau \rightarrow \tau + 1$ is a “full cigar” which has been bent such that the tips of the cigar touch.

In order to better understand the nongeometric aspects of these compactifications, consider a degenerate fiber around which $\rho \rightarrow -1/\rho$. More explicitly, after traversing a closed loop, one has a new metric and B-field with components

$$B'_{12} = \frac{-B_{12}}{B_{12}^2 + V^2} \quad , \quad V' = \frac{V}{B_{12}^2 + V^2} \quad .$$

It is clear that what was previously thought to be the B-field, B_{12} , and what was previously thought to be geometrical, V , now mix in a nontrivial way. Moreover, the modulus corresponding to rescaling ρ is fixed. This is easy to see: Let $\rho' \equiv \lambda\rho$. As one encircles the degenerate fiber, $\rho' \rightarrow \lambda/\rho \neq -1/\rho'$. The boundary conditions imposed by the monodromy are no longer satisfied. One is not free to rescale the T^2 ; the volume modulus of the fiber is no longer present in these compactifications.

One may gain further insight into these spaces by considering an analogy with classical geometry. The orbifold T^4/\mathbb{Z}_2 has 16 fixed points which may be smoothed out by replacing the local geometry near each fixed point with an Eguchi-Hansen space; this is the Kummer construction of a K3 manifold. The radii \mathbf{r}_i of each individual Eguchi-Hansen space are moduli describing the K3. The orbifold T^4/\mathbb{Z}_2 is then seen as a particular limit in the K3 moduli space where $\mathbf{r}_i \rightarrow 0$ for all i . It is said that the K3 is the “blow up” of the T^4/\mathbb{Z}_2 orbifold. Analogously, HMW show that there exists a special point in the moduli space of these nongeometric compactifications which corresponds to an asymmetric orbifold. This limit is discussed in detail below. Since there is a such a point in the moduli space of these nongeometric spaces which corresponds to asymmetric orbifolds, and since there is a strong similarity between the construction of these spaces and K3 manifolds, it is natural to think of these nongeometric spaces as “blow ups” of asymmetric orbifolds.

2.2. The $12 + 12'$ Orbifold Limit

Let us now take a closer look at the orbifold point discussed above. The $12 + 12'$ orbifold limit is

$$(T^3 \times \mathbf{R})/\mathbb{Z}_2 \times \mathbb{Z}'_2 \quad \text{with} \quad (2.7)$$

$$\mathbb{Z}_2 = \mathcal{I}_4 \quad \text{and} \quad \mathbb{Z}'_2 = (-1)^{F_L} P_\Delta \mathcal{I}_4 \quad .$$

Here \mathcal{I}_4 is reflection on all the coordinates of the T^3 as well as the noncompact coordinate x , F_L is left-moving spacetime fermion number and P_Δ is a translation by a distance Δ along the noncompact direction.

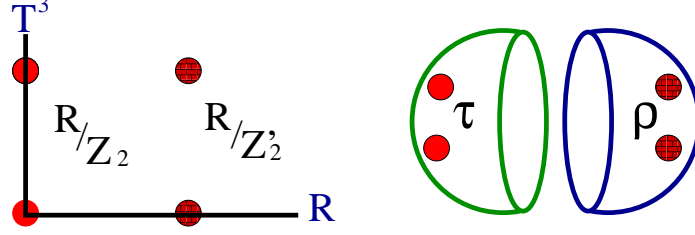


Fig. 1: The 8 fixed points at $x = 0$ are resolved τ -degenerations, and the 8 fixed points at $x = \Delta$ are resolved ρ -degenerations.

Associated with the \mathbb{Z}_2 are 8 fixed points located at $x = 0$. The twisted sector states localized at these fixed points are the same as the states in the twisted sector of IIA on $\mathbf{R}^4/\mathcal{I}_4$. In addition, there are another 8 fixed points, which come from \mathbb{Z}'_2 , located at $x = \Delta$. The twisted states on each of these \mathbb{Z}'_2 fixed points are the same as the twisted states of IIA on $\mathbf{R}^4/\mathcal{I}_4(-1)^{F_L}$. As discussed above, K3 has a orbifold limit in which all 16 fixed points are described by $\mathbf{R}^4/\mathcal{I}_4$. The 8 \mathbb{Z}_2 fixed points in the $12 + 12'$ model are identical to those in the orbifold of Type IIA on a K3. Moreover, since orbifolding type IIA by $(-1)^{F_L}$ gives type IIB, the \mathbb{Z}'_2 fixed points are identical to those in the orbifold limit of Type IIB on a K3. This should not be surprising, since the $12 + 12'$ model was constructed by building a compact space with 12 τ degenerations and 12 ρ degenerations. Since T-duality exchanges τ degenerations in type IIB with ρ degenerations in type IIA, one can then think of the $12 + 12'$ model as gluing together half of a K3 in type IIA with the T-dual of half of a K3 in type IIB (see fig. 1).¹

It turns out that the $12 + 12'$ orbifold can be reformulated as an asymmetric orbifold of the K3 orbifold limit. First note that (2.7) can be rewritten as

$$\begin{aligned} & ((T^3 \times \mathbf{R})/\mathbb{Z}_2'')/\mathbb{Z}_2 \quad \text{with} \\ & \mathbb{Z}_2 = \mathcal{I}_4 \quad \text{and} \quad \mathbb{Z}_2'' = (-1)^{F_L} P_\Delta . \end{aligned} \tag{2.8}$$

Moreover,

$$\begin{aligned} & ((T^3 \times \mathbf{R})/\mathbb{Z}_2'') = ((T^3 \times S_{2\Delta}^1)/\mathbb{Z}_2''') \\ & \text{where } \mathbb{Z}_2''' = (-1)^{F_L} s . \end{aligned} \tag{2.9}$$

Here $S_{2\Delta}^1$ is a circle of radius 2Δ and s is a half shift around the S^1 . The T^3 and the S^1 combine to give a T^4 , so we find that (2.7) can be expressed as

$$(T^4/\mathbb{Z}_2''')/\mathbb{Z}_2 = (T^4/\mathbb{Z}_2)/\mathbb{Z}_2''' . \tag{2.10}$$

¹ This should not be confused with the “half-K3” spaces in [10].

Since the $12 + 12'$ orbifold point can be expressed as a freely acting asymmetric orbifold of the K3 orbifold point, it is natural to ask if the $12 + 12'$ models are simply freely acting asymmetric orbifolds of the K3. The authors of this paper are currently investigating this issue.

3. Type II Monodrofolds with a T^2 Base

In the remainder of the paper we will discuss new compactifications similar to those discussed in [2]. These compactifications will have a T^2 , rather than S^2 , base. Since the T^2 has non-trivial cycles it is no longer necessary to have singularities in the base; this dramatically simplifies the above construction. Due to the central role of monodromy in the constructions and in the backgrounds discussed in [2], we have chosen to refer these backgrounds as “monodrofolds.”²

In section 2 we briefly discussed supergravity solutions of the type II string compactified on a T^2 . Let us recall some of the relevant facts: The supersymmetry transformations of the gravitino $\Psi_{\mu\alpha}$ and dilatino λ_α in type II supergravity are [11,12],

$$\delta\lambda = (\Gamma_{[10]}\Gamma^\mu\partial_\mu\Phi - \frac{1}{6}\Gamma^{\mu\nu\sigma}H_{\mu\nu\sigma})\eta = 0 \quad (3.1)$$

$$\delta\Psi_\mu = (\partial_\mu + \frac{1}{4}\Omega_\mu^{\underline{MN}}\Gamma^{\underline{MN}})\eta \equiv \tilde{\nabla}_\mu\eta = 0 \quad . \quad (3.2)$$

As usual, we set the expectation values of fermionic fields to zero to obtain a Lorentz invariant six-dimensional theory, so we need not consider the SUSY variations of the bosonic fields. These constraints, for a T^2 fibered over an S^2 , were solved in the appendix of [2]. In the interest of completeness, the solutions to (3.1) and (3.2) for a T^2 base have been derived in appendix A. It turns out that the solutions carry over to the T^2 base almost without alteration: From the dilatino variation it follows that

$$\frac{\bar{\partial}B_{12}}{V} = i\chi_6\bar{\partial}\Phi \quad . \quad (3.3)$$

The gravitino variation along the fiber, $\delta\Psi_I$, reduces to the holomorphy³ of ρ and τ ,

$$\bar{\partial}\rho = 0 \quad \text{and} \quad \bar{\partial}\tau = 0 \quad , \quad (3.4)$$

² Just in case it is not clear, this is a heterosis of the words monodromy and manifolds, monodro(my)(mani)folds.

³ Note that one must replace the complex coordinate z on the sphere with the complex coordinate on the torus, $z = \tau\tilde{\theta}_1 + \tilde{\theta}_2$.

and the gravitino variation along the base, $\delta\Psi_i$ becomes

$$0 = \partial\bar{\partial}(\ln \tilde{\rho}_2 - \ln \rho_2 - \ln \tau_2) . \quad (3.5)$$

Here τ and ρ are defined as in (2.2) and (2.3). Similarly, we will denote the complex structure and Kähler moduli of the base with $\tilde{\tau}$ and $\tilde{\rho}$. In general, we will always use tildes to denote parameters describing the base.

3.1. The Metric, B-field, and Ramond-Ramond Moduli

Unlike the $12 + 12'$ models constructed in [2], which had a singular S^2 base, the monodrofold we are discussing have a nonsingular T^2 base. This is a tremendous simplification. Indeed, as we will show below, this now tells us that ρ and τ must be constant. It follows that the only monodromies allowed are those with fixed points! The simplest example is

$$\begin{aligned} \rho &\rightarrow \rho , \quad \tau \rightarrow -1/\tau \quad (\tilde{\theta}^1 - cycle) \\ \rho &\rightarrow -1/\rho , \quad \tau \rightarrow \tau \quad (\tilde{\theta}^2 - cycle). \end{aligned} \quad (3.6)$$

Since ρ and τ must be constant it follows that $\rho = \tau = i$. All of the fiber moduli are fixed by the monodromy (3.6). On the surface this compactification looks very much like $T^2 \times T^2$, where the fiber T^2 is square and there is no B-field. However due to the boundary conditions (3.6) there do not exist any massless six-dimensional fields coming from moduli of the T^2 fiber.

Let us now show why the fields ρ and τ must lie at fixed points of the monodromy. It follows from (3.5) that if one allows the moduli of the fiber to vary there is a nontrivial backreaction on the base. Such a backreaction is not possible on a T^2 . Let us be more explicit. The metric on the base in complex coordinates takes the form

$$ds_{base}^2 = \frac{\tilde{\rho}_2(z, \bar{z})}{\tilde{\tau}_2} |\tilde{\tau} d\tilde{\theta}_1 + d\tilde{\theta}_2|^2 \equiv \frac{\tilde{\rho}_2(z, \bar{z})}{\tilde{\tau}_2} |dz|^2 , \quad (3.7)$$

this is just the usual metric written in terms of the moduli $\tilde{\tau}$ and $\tilde{\rho}$. It follows that Ricci scalar R is a total derivative:

$$\begin{aligned} R &= -\nabla_{base}^2 \ln \tilde{\rho}_2 = -\frac{\tilde{\tau}_2}{\tilde{\rho}_2} \partial\bar{\partial} \ln \tilde{\rho}_2 \\ &= -\frac{\tilde{\tau}_2}{\tilde{\rho}_2} \partial\bar{\partial} \ln \rho_2 \tau_2 = -\nabla_{base}^2 \ln \rho_2 \tau_2 . \end{aligned} \quad (3.8)$$

Going from the first line to the second line of (3.8) we have used (3.5). The requirement that the base be a T^2 forces the Euler characteristic to vanish;

$$\chi_{base} = \int_{T^2_{base}} R = 0 . \quad (3.9)$$

Since R is a total derivative this is trivially satisfied in compactifications where ρ_2 and τ_2 are single valued. However, except in the trivial case where ρ and τ are constants and lie at fixed points of the monodromy, if τ or ρ undergoes a nontrivial monodromy the surface term does not vanish. We can now see that in order to satisfy (3.9) and have a nontrivial monodromy, ρ and τ must be constant.

From the supergravity perspective the point is clear: When performing a dimensional reduction with a given background, one must make a choice for the boundary conditions. In addition to the traditional choice where the fields are periodic, one may choose boundary conditions in which the fields undergo a nontrivial monodromy. The choice of boundary conditions manifests itself in the field content of the dimensionally reduced theory, and each different dimensionally reduced theory should have a consistent world sheet description. In particular, for the monodrofolds with a T^2 base, we believe that they should be described by interpolating asymmetric orbifolds [13,14]. These orbifolds will be discussed in more detail below. Indeed, this is the conclusion reached in [1]. There, the authors use a Scherk-Schwarz argument to prove that stable backgrounds that include twists in the U-duality group must lie at fixed points of this group, and should have CFT descriptions as (asymmetric) orbifolds.

As will be discussed in Section 5, the construction of consistent asymmetric orbifolds is often a laborious exercise [13,14,15]. In contrast, the construction of monodrofolds is substantially simpler. The monodrofold construction has the additional advantage that it provides a spacetime interpretation of a certain class of asymmetric orbifolds. Note, however, that this “spacetime interpretation” is necessarily six dimensional; since one does not specify the boundary conditions until one performs a dimensional reduction it does not make sense to discuss these backgrounds from the higher dimensional perspective.

In addition to ρ and τ , one must also consider the moduli coming from the Ramond-Ramond sector. For the type IIA string compactified on a T^4 , the RR vector $c_{(1)}$ and the RR three form $c_{(3)}$ each give rise to four scalars in six dimensions. In the monodrofold discussed above, half of the corresponding moduli are lifted. To see this one must first ask how the potentials $c_{(1)}$ and $c_{(3)}$ transform. An $SL(2, \mathbf{Z})_\tau$ transformation simply mixes

the coordinates of the torus, and should therefore mix the potentials coming from branes wrapped on 1-cycles of the fiber T^2 . Such an operation would clearly not affect branes which wrap both cycles of the torus or branes which do not wrap the torus at all. We can see that the $SL(2, \mathbf{Z})_\rho$ action works similarly. Consider for example the monodromy $\rho \rightarrow -1/\rho$. This corresponds to T-dualizing both cycles of the fiber torus and performing a ninety degree rotation. The ninety degree rotation simply removes the ninety degree rotation induced by the two T-dualities. The T-dualities exchange branes which do not wrap any cycle of the T^2 with branes that wrap the whole torus.

In particular, the $SL(2, \mathbf{Z})_\tau$ monodromy, \mathcal{T} , acts on on the vectors

$$\begin{pmatrix} c_{\theta^1} \\ c_{\theta^2} \end{pmatrix} \text{ and } \begin{pmatrix} c_{\theta^1 \tilde{\theta}^1 \tilde{\theta}^2} \\ c_{\theta^2 \tilde{\theta}^1 \tilde{\theta}^2} \end{pmatrix} \quad (3.10)$$

via usual matrix multiplication and the $SL(2, \mathbf{Z})_\rho$ monodromy, \mathcal{T}' , acts on on the vectors

$$\begin{pmatrix} c_{\theta^1 \theta^2 \tilde{\theta}^1} \\ c_{\tilde{\theta}^1} \end{pmatrix} \text{ and } \begin{pmatrix} c_{\theta^1 \theta^2 \tilde{\theta}^2} \\ c_{\tilde{\theta}^2} \end{pmatrix} \quad (3.11)$$

in the same way. For definiteness, consider the 2-component object c_I ; this may be either one of the objects in (3.10). In the T^4 compactification there exist two six-dimensional moduli λ^i which correspond to rescalings of the c_I . Asking if the moduli λ^i are present in these compactifications is equivalent to asking if

$$\left[\begin{pmatrix} \lambda^1 & 0 \\ 0 & \lambda^2 \end{pmatrix}, \mathcal{T} \right] = 0 . \quad (3.12)$$

In particular, for the monodromy (3.6)

$$\mathcal{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

It follows that $\lambda^1 = \lambda^2$; there is a single modulus corresponding to an overall rescaling of the 2-vector c_I . Similar arguments hold for each of the vectors in (3.10) and (3.11). Thus, this boundary condition projects out half of the moduli coming from the RR sector.

4. Heterotic Monodrofolds with a T^2 base

The constraints arising from the the dilatino variation (3.1) and the gravitino variation (3.2) are the same in the heterotic string. There is an additional constraint arising from the variation of the sixteen gauginos,

$$\delta\gamma^{\mathcal{I}} = F_{\underline{\mathbf{MN}}}^{\mathcal{I}} \Gamma^{\underline{\mathbf{MN}}} \eta = 0. \quad (4.1)$$

Since we are projecting onto spinors of definite four dimensional chirality χ_4 on the compact space, (4.1) reduces to the condition that F_2 be (anti)self-dual⁴,

$$0 = F_{\underline{I}}^{\underline{AB}} - \frac{\chi_4}{2} \epsilon^{\underline{ABCD}} F_{\underline{I}}^{\underline{CD}} . \quad (4.2)$$

Converting to spacetime indices and noting that $\partial_I A_{\underline{I}} = 0$ (since we have assumed that nothing depends on the coordinate along the fiber), this becomes $F_{IJ}^{\mathcal{I}} = F_{ij}^{\mathcal{I}} = 0$ and

$$0 = \partial_i A_J^{\mathcal{I}} + \chi_4 \epsilon_{ij} \epsilon_{JK} \partial^j A_{\mathcal{I}}^K . \quad (4.3)$$

Note that although this is naïvely true only for Abelian gauge fields, we can always choose the Wilson lines to be in a $U(1)^{16}$ subgroup of either $E_8 \times E_8$ or $SO(32)$; this is due to the fact that the Wilson lines in compact directions pick up a potential proportional to the square of their commutator when dimensionally reduced. For generic backgrounds, the only solution to (4.3) is $\partial_i A_J^{\mathcal{I}} = 0$. Thus, we find that the gaugino variation requires that the gauge fields along the base are flat and the gauge fields along the fiber are constant.

4.1. Conventions

Keeping track of indices becomes even more messy in the heterotic string than in the type II string. In an attempt to simplify the equations, we will work in matrix notation. As we will describe in detail in the next subsection, we will discuss the heterotic string in terms of its embedding into the 26-dimensional bosonic string. Our main concern will be transformations of the background matrix

$$E \equiv G + B .$$

The rows and columns of E run over the entire compact space (base and fiber) as well as the internal directions associated with the 16 left-moving bosons. We will break the background matrix E into blocks,

$$E = \begin{pmatrix} E_{bb} & E_{bf} & E_{bI} \\ E_{fb} & E_{ff} & E_{fI} \\ E_{Ib} & E_{If} & E_{II} \end{pmatrix} . \quad (4.4)$$

⁴ As discussed in Appendix A, $\underline{\mathbf{A}}$ etc. are tangent space indices along both the base and the fiber. We also use I as the spacetime index along the fiber, i as the spacetime index along the base, and $\underline{\mathbf{I}}$ as the spacetime index along both.

The subscripts b , f and I stand for “base”, “fiber” and “internal” respectively. The subscripts simply label the directions along which the row and column of a given matrix point. They should not be confused with indices: they are not raised and lowered with a metric; repeated labels are not necessarily summed. The transpose of a matrix is denoted by a superscript T . When it is necessary to include indices we will label the internal directions with indices $\mathcal{I}, \mathcal{J}, \mathcal{K}, \dots$ and let $\underline{\mathcal{I}}, \underline{\mathcal{J}}, \underline{\mathcal{K}}, \dots$ run over the fiber, base and internal directions.

4.2. Heterotic T -duality group

Recall that perturbative duality group for the type II string compactified on a T^2 is

$$\mathcal{G}_{II} \equiv O(2, 2; \mathbb{Z}) \sim SL(2, \mathbb{Z})_\tau \times SL(2, \mathbb{Z})_\rho . \quad (4.5)$$

The decomposition of \mathcal{G}_{II} into $SL(2, \mathbb{Z})$ subgroups naturally lent itself to a description in terms of elliptic fibers and the tools developed in [9]. Such a description was integral to the construction in [2]. In what follows we are interested in the T -duality group of the heterotic string compactified on a T^4 (more appropriately, a T^2 fibered over a T^2);

$$\mathcal{G}_{HET} \equiv O(20, 4; \mathbb{Z}) , \quad (4.6)$$

is much more complicated than its type II counterpart \mathcal{G}_{II} . This structure has made the heterotic string extremely interesting from a phenomenological standpoint. However, for our purposes in this work, the larger duality group significantly complicates matters: The duality group is no longer just a product of two $SL(2, \mathbb{Z})$ groups. In the work that follows we focus on a subgroup of (4.6) which acts only on the fiber directions.

The heterotic string may be embedded into the 26-dimensional bosonic string in the following manner (for a review see [16]): The bosonic part of the action is simply

$$S_{HET} \sim \int d^2z E_{\underline{\mathcal{I}}\underline{\mathcal{K}}} \partial X^{\underline{\mathcal{I}}} \bar{\partial} X^{\underline{\mathcal{K}}} , \quad (4.7)$$

with the constraint that $X^{\mathcal{I}}$ be chiral bosons (since the internal directions are only along one side of the string); this must be added by hand. Moreover, the background matrix E must take the following form,

$$E = \begin{pmatrix} E_{bb} & E_{bf} & E_{bI} \\ E_{fb} & E_{ff} & E_{fI} \\ E_{Ib} & E_{If} & E_{II} \end{pmatrix} \equiv \begin{pmatrix} (G + B + \frac{1}{4}A^2)_{bb} & \frac{1}{4}A_{bf}^2 & A_{bI} \\ \frac{1}{4}A_{fb}^2 & (G + B + \frac{1}{4}A^2)_{ff} & A_{fI} \\ 0 & 0 & E_{II} \end{pmatrix} . \quad (4.8)$$

The internal background E_{II} is associated with the cartan matrix $C_{\mathcal{IJ}}$ of the enhanced symmetry group of the heterotic string. Namely,

$$E_{\mathcal{IJ}} = 0 \ (\mathcal{I} < \mathcal{J}); \ E_{\mathcal{II}} = 1 \ (\text{NO SUM}); \ E_{\mathcal{IJ}} = C_{\mathcal{IJ}} \ (\mathcal{I} > \mathcal{J}) \ .$$

As reviewed in [16], this is a consequence of the Narain lattice including the weight lattice of the gauge group. The indices we have been using are actually in a dual lattice, which then contains the root lattice of the gauge group; the natural metric on the root lattice is the Cartan matrix.

One may view this as an compactification on a background where

$$G_{\underline{\mathcal{I}}\underline{\mathcal{I}}} = B_{\underline{\mathcal{I}}\underline{\mathcal{I}}} \implies G_{\mathcal{I}\underline{\mathcal{I}}} = -B_{\mathcal{I}\underline{\mathcal{I}}} \ .$$

It is easy to check that dimensional reduction on the internal space gives the 10 dimensional heterotic action with Wilson lines $A_{\underline{\mathcal{I}}}^{\mathcal{I}}$. For the heterotic string compactified on a T^2 , the \mathcal{G}_{HET} transformations may now be described as $O(20, 20; \mathbb{Z})$ transformations of the background (4.8). More specifically, the $O(20, 20; \mathbb{Z})$ symmetry group may be generated by integral shifts of the B-field,

$$B_{ij} \rightarrow B_{ij} + \Theta_{ij} \quad (\Theta_{ij} = -\Theta_{ji} \in \mathbb{Z}) \ , \quad (4.9)$$

basis changes of the compactification lattice,

$$E \rightarrow MEM^T \quad (M \in O(20, \mathbb{Z})) \ , \quad (4.10)$$

and T-duality. This is reviewed in great detail in [16]. As in the type II case we will concentrate on examples in which the T^2 fiber undergoes these operations.

4.3. Heterotic Monodrofolds

We are now in a position to give examples of heterotic monodrofolds with a T^2 base. As mentioned above we will focus our attention on monodrofolds in which the fiber alone undergoes monodromies. Shifts in the B-field and the change of basis have already been made explicit in the previous subsection. For inversion, one may follow the steps outlined in [16] to show that

$$\begin{aligned} E_{ff} &\rightarrow E_{ff}^{-1} \\ A_{fI} &\rightarrow -E_{ff}^{-1} A_{fI}. \end{aligned} \quad (4.11)$$

This inversion also acts on the other components of E and A , but this action is trivial in the case where $A_{fI} = 0$. As it turns out, this will always be the case in our examples. Since we are only considering transformations along the fiber directions we will drop the “fiber-fiber” (ff) subscripts in what follows. Keep in mind that the background E being discussed is a 2×2 matrix. This simplifies much of the algebra, and many of the formulae which follow are special to the 2×2 case. This being said, generalizing to other backgrounds and transformations should introduce nothing more than tedium.

In the remainder of this subsection, we will concentrate on examples in which the above three operations have been performed at most once. Thus, in this case there are really only three different operations we need consider (all other possibilities are equivalent to special cases of these):

$$\begin{aligned} 1) \quad & E \rightarrow ME^{-1}M^T + \Theta \\ 2) \quad & E \rightarrow M(E + \Theta)^{-1}M^T \\ 3) \quad & E \rightarrow (MEM^T + \Theta)^{-1}. \end{aligned} \tag{4.12}$$

Note that this is still only a small subset of the $O(20, 20, \mathbb{Z})$ monodromies which may be imposed.

We will now go through one example in detail, that of the monodromy $E \rightarrow M(E + \Theta)^{-1}M^T$. First note that in this case, the action on the gauge field along the fiber is $A \rightarrow -(E + \Theta)^{-1}A$, which is a consequence of composing the three $O(20, 20, \mathbb{Z})$ transformations in the particular order we have chosen for this example. Recall that the constraints arising from supergravity force the matrix E_{ff} as well as the gauge fields A_{fI} , along the fiber, to lie at fixed points of the monodromy. Thus, we require

$$\begin{aligned} E &= M(E + \Theta)^{-1}M^T \\ A &= -(E + \Theta)^{-1}A. \end{aligned} \tag{4.13}$$

It turns out that it is possible to make some general comments, without specifying the matrices M or Θ . We now run through these arguments in detail.

Taking the trace of the first equation in (4.13), we notice that $\det(E + \Theta)^{-1} = 1$. This is easy to see: $\text{Tr}E = \text{Tr}(E + \Theta)^{-1} = \text{Tr}E / \det(E + \Theta)$. Thus, either $\text{Tr}E = 0$ or $\det(E + \Theta) = 1$. Writing out the matrix E ,

$$E_{ff} = \begin{pmatrix} G_{11} + A_{11}^2 & G_{12} + A_{12}^2 + B_{12} \\ G_{12} + A_{12}^2 - B_{12} & G_{22} + A_{22}^2 \end{pmatrix}, \tag{4.14}$$

we see that $\text{Tr}E = G_{11} + G_{22} + A_{11}^2 + A_{22}^2 > 0$, since both G_{11} and G_{22} must be positive definite; this is a result of requiring $\det G > 0$ and our ansatz that the off-diagonal terms in the metric (between the fiber and the base) vanish. Thus, $\det(E + \Theta) = 1$, which also implies $\det E = 1$ as a result of our monodromy.

Now, notice that the second equation in (4.13) implies that, for nonzero A_{fI} , the matrix $(E + \Theta)^{-1}$ has at least one eigenvalue equal to -1 . Since basis changes in the orthogonal group (4.10) preserve eigenvalues, $E = M(E + \Theta)^{-1}M^T$ implies that $(E + \Theta)^{-1}$ and E have the same eigenvalues. Thus E must also have at least one eigenvalue equal to -1 . But since $\det E = 1$, the other eigenvalue must then be -1 , implying $\text{Tr}E < 0$. However, since the fiber is purely spacelike this is not possible. It follows that the gauge field along the fiber must vanish; $A_{fI} = 0$.

We can now impose some restrictions on the B field along the fiber: Using

$$\det(E + \Theta) = 1 + \text{Tr}E^{-1}\Theta + \det E^{-1}\Theta = 1, \quad (4.15)$$

one can see that $\text{Tr}E^{-1}\Theta + \det \Theta = 0$. Writing

$$\Theta = \begin{pmatrix} & -\theta \\ \theta & \end{pmatrix}, \quad \theta \in \mathbb{Z} \quad (4.16)$$

and noting that

$$E^{-1}\Theta = \theta \begin{pmatrix} -G_{12} - B_{12} & -G_{22} \\ G_{11} & G_{12} - B_{12} \end{pmatrix}, \quad (4.17)$$

we get $-2B_{12}\theta + \theta^2 = 0$, implying that $B_{12} = \theta/2$. Thus, in the presence of nonzero θ , the moduli for the B -field along the fiber are lifted.

What about moduli for G ? It turns out that, for generic M with $\det M = 1$, all the moduli in the fiber metric are also fixed by our monodromy. The calculation is straightforward, and yields

$$G = \frac{\sqrt{4 - \theta^2}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.18)$$

This implies that $|\theta| < 4$, which means the only possibilities are $\theta = -1, 0, 1$. For generic M with $\det M = -1$, there are no solutions; such transformations are thus not allowed.

Although these results are true in general, there are special cases of M where the above is incorrect. Taking

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (4.19)$$

we find that $B_{12} = \theta/2$, and the metric is arbitrary (up to the condition $\det E = 1$).

For

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.20)$$

we find solutions only if $\theta = 0$. First,

$$G = \begin{pmatrix} 1 & G_{12} \\ G_{12} & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & G_{12} \\ -G_{12} & 0 \end{pmatrix}. \quad (4.21)$$

The second solution for this M is

$$G = \begin{pmatrix} G_{11} & \pm\sqrt{B_{12}^2 + G_{11}^2 - 1} \\ \pm\sqrt{B_{12}^2 + G_{11}^2 - 1} & G_{11} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B_{12} \\ -B_{12} & 0 \end{pmatrix}, \quad (4.22)$$

with B_{12} and G_{11} arbitrary, subject to the condition that $G_{11}^2 + B_{12}^2 - 1 \geq 0$.

Finally, for

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (4.23)$$

we find that the only solutions are those with $\theta = 0$:

$$G = \begin{pmatrix} G_{11} & 0 \\ 0 & \frac{1-B_{12}^2}{G_{11}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B_{12} \\ -B_{12} & 0 \end{pmatrix} \quad (4.24)$$

with B_{12} and G_{11} arbitrary.

The fixed points of the other two monodromies in (4.12) can be figured out in a similar fashion. It turns out that the generic solutions are in fact identical to what we just found: $A = 0$, $B = \theta/2$, and G is as in (4.18). This statement is not *a priori* obvious, and is apparently a consequence of the monodromies we have chosen to study.

5. Worldsheet Constructions

Throughout this paper we have worked from the spacetime perspective, exploiting tools from supergravity, in order to obtain new string theory backgrounds. If these are indeed legitimate string backgrounds one should be able to describe these compactifications in terms of a consistent worldsheet conformal field theory. It is often very difficult to find the CFT corresponding to a given string theory background. However, we believe that there is a strong indication that the monodromies with a T^2 base discussed in the previous two sections can be realized as interpolating orbifolds; similar conclusions are reached by the authors of [1]. To briefly review, an interpolating orbifold simply combines any order n discrete transformation g of the worldsheet theory with an order n geometric

shift s along a compact spacetime coordinate to form a single order n orbifold element gs ⁵. If we start with a theory H on \mathbf{R} then H on S^1/gs interpolates between the parent theory H on \mathbf{R} (as $R \rightarrow \infty$) and the pure orbifold H/g on $S^1_{R/n}$ (as $R \rightarrow 0$). Note that the monodromies discussed above have a similar interpolating behavior. Imposing a particular monodromy gives a mass to various fields in the theory, which is inversely proportional to the volume of the base T^2 . In the large volume limit, the fields become effectively massless and we are left with a standard T^2 compactification. As the volume of the base goes to zero, these fields become infinitely massive and get projected out of the theory completely. For a given monodromy, finding the corresponding interpolating orbifold boils down to identifying the discrete transformation of the worldsheet theory corresponding to the particular monodromy of interest.

When the monodromy element corresponds to a nongeometric transformation of the spacetime theory, we must consider nongeometric orbifold elements. In many cases these are transformations acting asymmetrically on the left and right movers on the worldsheet. The simplest example of this type is the monodromy $\rho \rightarrow -1/\rho, \tau \rightarrow -1/\tau$ which corresponds to a T-duality transformation on both cycles of the T^2 fiber. We know that from the worldsheet perspective T-duality is realized as a reflection on the left-moving fields. One would naïvely guess that such a monodromy could be described from the worldsheet perspective as

$$T^2 \times S^1_{2R}/\mathbb{Z}_2, \quad \mathbb{Z}_2 = \mathcal{I}_2^L s. \quad (5.1)$$

Here \mathcal{I}_2^L is a reflection on left hand side of the T^2 embedding coordinates, and s is a half-shift of the S^1 . As will be discussed below, this is not quite right.

The consistency of interpolating asymmetric orbifolds follows from the consistency of both the parent theory H and the pure orbifold H/g . Unfortunately, the consistency of even pure (i.e. not interpolating) asymmetric orbifolds is a highly nontrivial subject. In the symmetric case, modular invariance is guaranteed because the theory is deformed identically on both sides of the worldsheet; this naïvely preserves the level-matching of ground states from the parent theory⁶. For an orbifold that acts differently on the left and right movers, one generically loses the level-matching of ground states. Thus, one

⁵ This is to be distinguished from $g \times s$.

⁶ Since the heterotic string is an asymmetric parent theory a symmetric orbifold may upset level matching. However, as long as we embed the spin connection in the gauge connection we are symmetrically deforming the same set of fields on each side of the worldsheet.

must work harder to find asymmetric orbifold actions which lead to modular invariant theories. For a given asymmetric transformation defined in the point group of the worldsheet theory, i.e. an asymmetric rotation or gauge rotation, one may try to doctor up the level mismatch by introducing pure translations on the left and right movers which include geometric and nongeometric coordinate shifts. Even allowing these modifications does not save most asymmetric rotations. For example, there is no way to gauge the rotation of a single left-moving embedding field (a single T-duality) for any parent theory and with any combination of shifts.

Let's return to the interpolating orbifold described in (5.1). The parent theory, T^2/\mathcal{I}_2^L , was shown not to be modular invariant in [15]. It was further shown the level mismatch could not be restored by the inclusion of geometric and nongeometric shifts. One might be tempted at this point to question the consistency of these models. However, at the level of supergravity we know that these backgrounds are consistent: recall that all we have done is taken a consistent solution and imposed consistent boundary conditions. If there does not exist a corresponding worldsheet CFT this would be an indication that the perturbative duality group is broken at the quantum level. The authors of this paper do not suspect that this is the case. A much more conservative conclusion is that although (5.1) was the simplest choice, there is another interpolating orbifold which gives the correct spacetime behavior and is modular invariant. Indeed, there are many other interpolating orbifolds in which the T^2 embedding coordinates are antiperiodic under $2\pi R$ shifts. The authors of this paper are currently investigating many of these possibilities.

6. Discussion

In this paper, we have used nontrivial boundary conditions to arrive at six dimensional theories with fewer moduli than their counterparts with periodic boundary conditions. Such nongeometric compactifications seem, at first glance, to be quite strange. One might question their relevance in describing “real world” physics. This being said, there are certainly no physical or theoretical reasons to suspect the compact dimensions to have a geometric interpretation. Moreover, as noted in [17,11] it is often the case that geometric backgrounds with fluxes are dual to nongeometric backgrounds. In fact, it has recently been pointed out that the Klebanov-Strassler solution [18], which has played a central role in the recent discussions of De Sitter vacua in string theory [19], has a nongeometric dual [20]. To show this, first note that the conifold with NSNS flux may be decomposed as a T^3

fibration over a noncompact three-dimensional base. The monodromy group of the fiber includes, in addition to geometric actions, monodromies which shift the periods of the B-field. T-dualizing along each of the fibers results in a dual theory in which the monodromies lie outside of the group generated by the geometric $SL(3, \mathbb{Z})$ transformations and B-field shifts. The new monodromies include nongeometric monodromies, i.e. monodromies which change volumes of the 2-cycles and the dimensionality of branes wrapped on various cycles.

It is becoming increasingly clear that nongeometric backgrounds are an indeed important ingredient of string theory. Such vacua, however, have not been thoroughly explored, leaving many open questions for future work. This work is a step in exploring these vacua. Of course, we have not here studied the most general nongeometric boundary conditions possible. In particular, our discussion of the heterotic string imposes monodromies which are a very small subset of the $O(20, 4, \mathbb{Z})$ duality group. We find that in general, these monodromies must have fixed points and that in all cases the Wilson lines along the fiber are forced to vanish. A natural question is to ask whether one can see this (from a space-time perspective) in more general nongeometric spaces. It would also be interesting to find backgrounds in which the Wilson lines are nonvanishing.

As mentioned in section 5, one would like to have a worldsheet description of nongeometric theories. A natural guess for such a description is that it is an interpolating asymmetric orbifold. However, the construction of asymmetric orbifolds is known to be a very delicate procedure [13,14,15]. Finding such a description would be a big step in understanding general nongeometric backgrounds. Another interesting question would be to find a string dual of the nongeometric compactifications in [2] or this paper. Since the $12 + 12'$ is very similar to an orbifold of K3 (see Section 2.2), it may be possible to do this using the techniques of [21]. One would suspect that the dual of the IIA theory would be some kind of nongeometric compactification of the heterotic string, possibly one of the models discussed here.

Another thing that one should note is that in this paper we have not checked that these nongeometric backgrounds are anomaly free. However, as discussed in Section 5, all we have done is taken a consistent theory and imposed consistent boundary conditions. We have no reason to suspect that, at the level of supergravity, this should result in an inconsistent theory. To better understand the nongeometric theories presented in this paper, it would be interesting to compute the spectrum and show explicitly that they are anomaly free.

Finally, we mention that in general, nongeometric compactifications of string theory are not well-understood. It may be possible to find such compactifications which are neither monodrofolds nor asymmetric orbifolds, but something else entirely. There are many possibilities for string vacua among such spaces, and by analogy with the nongeometric examples discussed here, one could hope that these also project out moduli. In order to obtain a more complete picture of string vacua, it is essential that we study such nongeometric spaces.

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Appendix A. Conventions

Unfortunately with the total space, the total compact space, the base space, the fiber space and all of the respective tangent spaces, it becomes quite ugly trying to keep track of all the different kinds of indices. (The situation is even worse for the heterotic string!) We will try our best to keep things clear. In general, boldfaced underlined letters correspond to the entire compact space. Lowercase letters go with the base and uppercase letters go with the fiber. Moreover, the letters at the beginning of the alphabet (A, B, C, ...) correspond to the tangent space indices and letters in the middle of the alphabet (I, J, K, ...) correspond to spacetime indices.

More explicitly we choose the following conventions for our coordinates: X^μ are the coordinates on all of spacetime. $\tilde{\theta}^i$ are the coordinates on the T^2 base, and θ^I are the coordinates on the T^2 fiber.

Let $\underline{\mathbf{M}}, \underline{\mathbf{N}}, \underline{\mathbf{P}}, \dots$ be the entire set of tangent space indices; let a, b, c, \dots be the tangent space indices corresponding to $\tilde{\theta}^i$; let A, B, C, \dots be the tangent space indices corresponding to θ^I ; let $\underline{\mathbf{A}}, \underline{\mathbf{B}}, \underline{\mathbf{C}}, \dots$ be indices that run over both a and A . We take the θ^I coordinates (and similarly for $\tilde{\theta}^i$) to have constant periodicity $\theta^I \sim \theta^I + n^I, n^I \in \mathbb{Z}$.

Additionally, tildes denote coordinates and fields parameterizing the base torus: $\tilde{\theta}^i, \tilde{\tau}, \tilde{\rho}, \tilde{b}, \tilde{V}$. $e_i^a (f_I^A)$ is the vielbein on the base (fiber). Finally, we use $\tilde{\nabla}$ to denote the generalized covariant derivative (i.e. the one corresponding to the generalized spin connection) acting on spinors and ∇ to denote the ordinary covariant derivative.

These conventions are consistent with [2], though we will make one small change. In [2] the generalized spin connection was defined to be

$$\Omega_\mu^{\underline{\text{MN}}} \equiv \omega_\mu^{\underline{\text{MN}}} + H_\mu^{\underline{\text{MN}}} \Gamma_{[10]} \quad (\text{OLD}) .$$

We will use the conventions of [22] where

$$\Omega_\mu^{\underline{\text{MN}}} \equiv \omega_\mu^{\underline{\text{MN}}} + \frac{1}{2} H_\mu^{\underline{\text{MN}}} \Gamma_{[10]} \quad (\text{NEW}) . \quad (\text{A.1})$$

This new definition of $\Omega_\mu^{\underline{\text{MN}}}$ effectively redefines the B-field such that $\rho \equiv b + iV$ is holomorphic (rather than $b + iV/2$). Here we are using $b \equiv B_{12}$ and $V \equiv \sqrt{\det M}$, rather than their respective periods, since it is these objects which change under the Buscher rules. This being said, in the compactifications we are considering these are constants, and since the torus coordinates have period = 1, B_{12} and $V \equiv \sqrt{\det M}$ are equal to their periods.

Appendix B. Type II Supergravity on a T^2

The supersymmetry transformations of the gravitino $\Psi_{\mu\alpha}$ and dilatino λ_α in type II supergravity are [11,12],

$$\delta\lambda = (\Gamma_{[10]} \Gamma^\mu \partial_\mu \Phi - \frac{1}{6} \Gamma^{\mu\nu\sigma} H_{\mu\nu\sigma}) \eta = 0 \quad (\text{B.1})$$

$$\delta\Psi_\mu = (\partial_\mu + \frac{1}{4} \Omega_\mu^{\underline{\text{MN}}} \Gamma_{\underline{\text{MN}}}) \eta \equiv \tilde{\nabla}_\mu \eta = 0 \quad (\text{B.2})$$

The constraints coming from (B.1) and (B.2) for a T^2 fibered over a S^2 were solved in the appendix of [2]. These constraints should carry over to the T^2 base, but for completeness we will now rederive some of these for this case.

The derivation of the constraint coming from the dilatino variation is exactly as for the S^2 case in [2]. Thus, we will simply remind the reader that, as stated in section II, (B.1) reduces to

$$\frac{\bar{\partial} b}{V} = i\chi_6 \bar{\partial} \Phi . \quad (\text{B.3})$$

As shown in [2], for a general two dimensional base \mathcal{B}_2 , the killing spinor equations along the fiber, $\tilde{\nabla}_I \eta = 0$, imply that ⁷

$$e^{ia} \partial_i V + \chi_6 \epsilon^{ab} e^{ib} \partial_i b = 0 \quad . \quad (\text{B.4})$$

For a spherical base $\mathcal{B}_2 = S^2$, we may take $e^{ia} \propto \delta^{ia}$, and these equations reduce to the Cauchy-Riemann equations, $\bar{\partial} \rho = 0$. Moreover, it can be shown $\bar{\partial} \tau = 0$. This can be seen directly, as in [2], or seen as a consequence of the fact that T-duality interchanges ρ and τ . For $\mathcal{B}_2 = T^2$, (B.4) again reduces to holomorphy of ρ . Note however that holomorphy is now with respect to the complex coordinates on the torus, $z = \tau \theta_1 + \theta_2$, $\bar{z} = \bar{\tau} \theta_1 + \theta_2$.

In order to solve the killing spinor equations along the base, $\tilde{\nabla}_i \eta = 0$, we first note that, for the ansatz and basis of gamma matrices used in [2],

$$\Gamma^{AB} = \epsilon^{AB} \sigma^3, \quad \Gamma^{ab} = \epsilon^{ab} \sigma^3, \quad \Omega_I^{AB} = \Omega_I^{ab} = \Omega_i^{aA} = 0.$$

Going to complex coordinates, we see that the equation $0 = [\tilde{\nabla}_z, \tilde{\nabla}_{\bar{z}}] \eta$ reduces to

$$\begin{aligned} 0 &= \partial \Omega_{\bar{z}}^{\underline{MN}} \Gamma^{\underline{MN}} - \bar{\partial} \Omega_z^{\underline{MN}} \Gamma^{\underline{MN}} \\ &= \partial [(\epsilon^{AB} f^{AI} \nabla_{\bar{z}} f_I^B) + (\epsilon^{ab} e^{ai} \nabla_{\bar{z}} e_i^b) + (\frac{\bar{\partial} b}{V})] \\ &\quad - \bar{\partial} [(\epsilon^{AB} f^{AI} \nabla_z f_I^B) + (\epsilon^{ab} e^{ai} \nabla_z e_i^b) + (\frac{\partial b}{V})] \end{aligned} \quad (\text{B.5})$$

Here we have used the fact that the vielbein along the fiber satisfies

$$\epsilon^{AB} \epsilon_J^I f^{AI} f^{BJ} = 2 \det[f^{AI}] = 2V^{-1} \quad .$$

We will now consider each term (minus its complex conjugate) individually. Before moving on, though, it will be useful to note that

$$\Gamma_{IJ}^K = \Gamma_{iJ}^k = \Gamma_{ij}^K = 0 \quad . \quad (\text{B.6})$$

Moreover, the amount of algebra may be reduced by rescaling f_I^A (and similarly for e_i^a) since

$$\epsilon^{AB} f^{AI} \nabla_{\bar{z}} f_I^B = \epsilon^{AB} \alpha^{AI} \nabla_{\bar{z}} \alpha_I^A \quad ;$$

⁷ Note that this differs from HMW by a factor of 1/2. This is due to (A.1).

here $\alpha_I^A \equiv h(z, \bar{z}) f_I^A$ for an arbitrary function $h(z, \bar{z})$. We have chosen to work with the explicit (rescaled) vielbein α_I^A :

$$\begin{aligned}\alpha_1^1 &= |\tau|^2 & \alpha_2^1 &= \tau_1 \\ \alpha_1^2 &= 0 & \alpha_2^2 &= \tau_2 \quad .\end{aligned}\tag{B.7}$$

It is easy to show that $\epsilon^{AB} f^{AI} \nabla_{\bar{z}} f_I^B = \epsilon^{AB} f^{AI} \bar{\partial} f_I^B$. From the holomorphy of τ it follows that

$$\partial(\epsilon^{AB} f^{AI} \bar{\partial} f_I^B) - \bar{\partial}(\epsilon^{AB} f^{AI} \partial f_I^B) = 2i \frac{\partial \tau \bar{\partial} \bar{\tau}}{|\tau - \bar{\tau}|^2} = -2i \partial \bar{\partial} \ln \tau_2 \quad .\tag{B.8}$$

Now consider the the second term on the RHS of (B.5). We are free to chose a gauge in which $\tilde{\tau}$ is constant. After doing this, $\epsilon^{ab} e^{ai} \nabla_{\bar{z}} e_i^b = -\epsilon^{ab} e^{ai} \Gamma_{\bar{z}i}^j e_j^b$. A little algebra yields

$$\partial(\epsilon^{ab} e^{ai} \nabla_{\bar{z}} e_i^b) - \bar{\partial}(\epsilon^{ab} e^{ai} \nabla_z e_i^b) = 2i \partial \bar{\partial} \ln \tilde{\rho}_2 \quad .\tag{B.9}$$

Finally, note that the forth term is greatly simplified by the holomorphy of ρ .

$$\partial\left(\frac{\bar{\partial} b}{V}\right) - \bar{\partial}\left(\frac{\partial b}{V}\right) = -2i \partial \bar{\partial} \ln \rho_2\tag{B.10}$$

Putting all of this together, one finds that

$$0 = \partial \bar{\partial} (\ln \tilde{\rho}_2 - \ln \rho_2 - \ln \tau_2) \quad .\tag{B.11}$$

As stated, this is the same solution as found in [2] for a spherical base.

References

- [1] A. Dabholkar and C. Hull, “Duality twists, orbifolds, and fluxes,” JHEP **0309**, 054 (2003) [arXiv:hep-th/0210209].
- [2] S. Hellerman, J. McGreevy and B. Williams, “Geometric constructions of nongeometric string theories,” arXiv:hep-th/0208174.
- [3] T. H. Buscher, “A Symmetry Of The String Background Field Equations,” Phys. Lett. B **194**, 59 (1987).
- [4] P. S. Aspinwall, B. R. Greene and D. R. Morrison, “Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory,” Nucl. Phys. B **416**, 414 (1994) [arXiv:hep-th/9309097].
- [5] A. Kumar and C. Vafa, “U-manifolds,” Phys. Lett. B **396**, 85 (1997) [arXiv:hep-th/9611007].
- [6] B. R. Greene, A. D. Shapere, C. Vafa and S. T. Yau, “Stringy Cosmic Strings And Noncompact Calabi-Yau Manifolds,” Nucl. Phys. B **337**, 1 (1990).
- [7] C. Vafa, “Evidence for F-Theory,” Nucl. Phys. B **469**, 403 (1996) [arXiv:hep-th/9602022].
- [8] A. Strominger, S. T. Yau and E. Zaslow, “Mirror symmetry is T-duality,” Nucl. Phys. B **479**, 243 (1996) [arXiv:hep-th/9606040].
- [9] K. Kodaira, Annals of Math. **77** (1963) 563; Annals of Math. **78** (1963) 1.
- [10] D. Morrison, “Half K3 surfaces,” Talk given at STRINGS 2002
- [11] K. Becker and K. Dasgupta, “Heterotic strings with torsion,” JHEP **0211**, 006 (2002) [arXiv:hep-th/0209077].
- [12] F. Giani and M. Pernici, “N=2 Supergravity In Ten-Dimensions,” Phys. Rev. D **30**, 325 (1984).
- [13] K. S. Narain, M. H. Sarmadi and C. Vafa, “Asymmetric Orbifolds,” Nucl. Phys. B **288**, 551 (1987).
- [14] K. S. Narain, M. H. Sarmadi and C. Vafa, “Asymmetric Orbifolds: Path Integral And Operator Formulations,” Nucl. Phys. B **356**, 163 (1991).
- [15] K. Aoki, E. D’Hoker and D. H. Phong, “On the construction of asymmetric orbifold models,” arXiv:hep-th/0402134.
- [16] A. Giveon, M. Porrati and E. Rabinovici, “Target space duality in string theory,” Phys. Rept. **244**, 77 (1994) [arXiv:hep-th/9401139].
- [17] S. Kachru, M. B. Schulz, P. K. Tripathy and S. P. Trivedi, “New supersymmetric string compactifications,” JHEP **0303**, 061 (2003) [arXiv:hep-th/0211182].
- [18] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities,” JHEP **0008**, 052 (2000) [arXiv:hep-th/0007191].

- [19] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, “De Sitter vacua in string theory,” *Phys. Rev. D* **68**, 046005 (2003) [arXiv:hep-th/0301240].
- [20] S. Hellerman, Princeton Journal Club Talk, December 2003
- [21] C. Vafa and E. Witten, “Dual string pairs with $N = 1$ and $N = 2$ supersymmetry in four dimensions,” *Nucl. Phys. Proc. Suppl.* **46**, 225 (1996) [arXiv:hep-th/9507050].
- [22] J. Polchinski, “String Theory. Vol. 2: Superstring Theory And Beyond,”